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Articles

2-Layer *k*-Planar Graphs Density, Crossing Lemma, Relationships And Pathwidth

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The 2-layer drawing model is a well-established paradigm to visualize bipartite graphs where vertices of the two parts lie on two horizontal lines and edges lie between these lines. Several beyond-planar graph classes have been studied under this model. Surprisingly, however, the fundamental class of k-planar graphs has been considered only for $k = 1$ in this context. We provide several contributions that address this gap in the literature.
First, we show tight density bounds for the classes of 2-layer k-planar graphs with $k \in \{2, 3, 4, 5\}$. Based on these results, we provide a Crossing Lemma for 2-layer k-planar graphs, which then implies a general density bound for 2-layer k-planar graphs. We prove this bound to be almost optimal with a corresponding lower bound construction. Finally, we study relationships between k-planarity and h-quasiplanarity in the 2-layer model and show that 2-layer k-planar graphs have pathwidth at most $k + 1$ while there are also 2-layer k-planar graphs with pathwidth at least $(k + 3)/2$.

Keywords: 2-layer graph drawing; *k*-planar graphs; density; crossing lemma; pathwidth; quasiplanar graphs

1. INTRODUCTION

Beyond-planarity is an active research area that studies graphs admitting drawings that avoid certain forbidden crossing configurations. Research on this subject has attracted considerable interest due to its theoretical appeal and due to the need of visualizing real-world non-planar graphs. An extensive review of the related literature can be found in [1, 2]. In particular, a great deal of attention has been captured by two important beyond-planar graph families. The *k-planar* graphs, with $k \geq 1$, for which the forbidden configuration is an edge crossing more than k other edges, and the *h-quasiplanar* graphs, with $h \geq 3$, for which the forbidden configuration is a set of h pairwise crossing edges. The study of these two families finds its origins in the 1960s [3, 4], when the question arose about the *density* of these graphs, that is, the maximum number of edges of graphs in these families.[1]

Many works have addressed this extremal graph theoretical question and established upper bounds for k-planar and h-quasiplanar graphs for various values of k and h. For small k and h, these upper bounds have been proven to be tight by lower bound constructions achieving the corresponding density. The most significant results include tight density bounds for 1-planar graphs [5] ($4n - 8$ edges), 2-planar graphs [5] ($5n - 10$ edges), 3-planar graphs [6, 7] ($5.5n - 20$) and 4-planar graphs [8] ($6n - 12$). For general k, the currently best upper bound is $3.81\sqrt{k}\,n$, which can be derived from the result of Ackerman [8] on 4-planar graphs and from the renowned Crossing Lemma [9]. For h-quasiplanar graphs, despite considerable efforts, a density upper bound that is linear in the number of vertices exists only for $h \leq 4$ [10–13]. Namely, a tight upper bound exists for simple 3-quasiplanar (for short, *quasiplanar*) graphs. Here, *simple* means that any two edges meet at at most one point, which is either a common endvertex or an internal point. For general h, only super-linear upper bounds exist [14–19] but a linear bound has been conjectured [17].

These two families have also been studied from other perspectives. A notable relationship is that every simple k-planar graph is also simple $(k+1)$-quasiplanar [20], for every $k \geq 2$. It is also known that every *optimal* 3-planar graph, namely one with the maximum possible number of edges ($5.5n - 20$), is also 3-quasiplanar. This latter result follows from a characterization of the optimal 3-planar graphs [21], which also exists for the optimal 1- and 2-planar graphs [5, 21]. Note that these characterizations do not directly yield recognition algorithms; in fact, recognizing (non-optimal) k-planar graphs is NP-complete for every $k \geq 1$ [22]. The complexity of recognizing h-quasiplanar graphs is still open for any $h \geq 3$.

Aside from these two major families, we mention the *fan-planar* graphs, in which no edge is crossed by two independent edges or by two adjacent edges from different directions [23–26], and the *RAC graphs*, in which the edges are poly-lines with few bends and crossings only happen at right angles [27–30]. These and other graph classes have been also investigated with respect to their density, recognition and relationship with other classes; see also the recent survey [1].

Beyond-planar classes have also been studied under additional constraints on the placement of the vertices. In the *outer model* [23, 31–36] every vertex is incident to the unbounded region of

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the drawing, while in the 2-*layer model* [24, 25, 34, 37] the vertices lie on two horizontal lines and every edge is a y-monotone curve. The latter model requires the graph to be bipartite, and the constraints on the placement of the vertices emphasize the bipartite structure. The 2-layer model naturally arises in the context of tanglegram layouts [38–40], social network analysis [41] and neural networks [42, 43]. We also remark that the 2-layer model lies at the core of the Sugiyama framework for general layered drawings [44, 45].

In [37], it was shown that 2-layer RAC graphs have at most $\frac{3}{2}n - 2$ edges and that this bound is tight, exploiting a characterization which also leads to an efficient recognition algorithm. Later, Didimo [34] observed that 2-layer 1-planar graphs are 2-layer RAC graphs, and that the optimal graphs in these two classes coincide. Thus, the tight bound of $\frac{3}{2}n - 2$ edges extends to 2-layer 1-planar graphs. For h-quasiplanar graphs, Walczak [46] provided a density upper bound of $(h-1)(n-1)$ edges, following from the fact that *convex bipartite geometric* h-quasiplanar graphs can be $(h-1)$-colored so that edges with the same color do not cross. For (3-)quasiplanar graphs, the $2n - 2$ bound can be improved to $2n - 4$ by observing that they are planar bipartite graphs [47, 48]. Since fan-planar graphs are also quasiplanar, this density bound holds for 2-layer fan-planar graphs, as well. Further, this bound is tight for both classes, since the complete bipartite graph $K_{2,n}$ is 2-layer fan-planar. Note that 2-layer fan-planar graphs have been characterized [24] and can be recognized when the graph is biconnected [24] or a tree [49]. Another property that has been investigated in the 2-layer model is the pathwidth. Namely, 2-layer fan-planar graphs have pathwidth 2 [49], while 2-layer graphs with at most c crossings in total have pathwidth $2c + 1$ [50]; note that both results can be extended to general layered graphs. More in general, pathwidth is an important graph structure parameter [51–54] that has for instance many applications in graph drawing [55–60].

Our Contribution. From the discussion above it is evident that, in the wide literature on the 2-layer model, the study of the central class of k-planar graphs is completely missing, except for the special case $k = 1$. In this paper, we make several contributions toward filling this gap. We provide tight density bounds for 2-layer k-planar graphs with $k \in \{2, 3, 4, 5\}$ in Section 3. Exploiting these bounds, we deduce a Crossing Lemma for 2-layer graphs in Section 4. This implies a density upper bound for general values of k. We then show a lower bound construction that is within a factor of 1/1.84 from the upper bound. Finally, in Section 5, we investigate two additional properties. First, we prove that 2-layer 2-planar graphs are 2-layer quasiplanar, as in the case where the vertices are not restricted to two layers [20]. For larger k, we show a stronger relationship, namely, every 2-layer k-planar graph is 2-layer h-quasiplanar for $h = \lceil \frac{2}{3}k + 2 \rceil$. Second, we demonstrate that 2-layer k-planar graphs have pathwidth at most $k + 1$, which is the first result of this type, since they may have a linear number of crossings and may not be fan-planar. We contrast this upper bound on the pathwidth by presenting a family of 2-layer k-planar graphs with pathwidth at least $(k + 3)/2$.

2. PRELIMINARIES

We assume familiarity with basic concepts from graph theory and graph drawing; see e.g. [61, 62].

The 2-layer model. A *bipartite graph* $G = (U \dot{\cup} V, E)$ is a graph with vertex subsets U and V, so that $E \subseteq U \times V$. A *topological* 2-*layer graph* G is a bipartite graph drawn in the plane so that the vertices in U and V are mapped to distinct points on two horizontal lines L_u and L_v, respectively, and the edges are mapped to y-monotone Jordan arcs. We denote the vertices in U and in V as $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$, respectively, in the order in which they appear in positive x-direction along L_u and L_v. It can be assumed that edges (u_i, v_j) and (u_k, v_ℓ) cross if and only if $i < k$ and $j > \ell$ or $i > k$ and $j < \ell$, as otherwise one of the two edges can be rerouted to avoid some crossing. This is equivalent to saying that the edges are drawn straight-line, which also means that G is *simple*, i.e. no two adjacent edges cross each other, and every two independent edges cross each other at most once.

We denote the number of vertices of G by $n = p + q$ and the number of edges in E by m. We call G k-*planar* if each edge is crossed at most k times, and h-*quasiplanar* if there is no set of h pairwise crossing edges. Further, we say that a bipartite graph G is 2-*layer* k-*planar* (h-*quasiplanar*) if there exists a topological 2-layer k-planar (resp. h-quasiplanar) graph whose underlying abstract graph is isomorphic to G.

While we will mainly discuss 2-layer k-planar graphs in this paper, the density of 2-layer quasiplanar graphs will be a useful tool in the following section. Since it is known that these graphs are planar [47, 48] and bipartite by definition, and since $K_{2,n}$ is 2-layer quasiplanar, the density of 2-layer quasiplanar graphs can be bounded as follows:

Theorem 2.1. [47, 48] An n-vertex 2-layer quasiplanar graph has at most $2n - 4$ edges for $n \geq 3$. Also, there exist infinitely many 2-layer quasiplanar graphs with n vertices and $2n - 4$ edges.

We point out that Theorem 2.1 also can be obtained from the argumentation for *convex bipartite geometric quasiplanar graphs* in [46].

Tree and path decomposition. A *tree decomposition* of a graph $G = (V, E)$ is a tree T on vertices $B_1, \ldots, B_n$ called *bags* such that the following properties hold:

1) each bag B_i is a subset of V,
2) $V = \bigcup_{i=1}^{n} B_i$,
3) for every edge $(u, v) \in E$, there exists a bag B_i such that $u, v \in B_i$ and
4) for every vertex v, the bags containing v induce a connected subtree of T.

If T is a path, we call T a *path decomposition*. The *width* of a tree decomposition T is the maximum cardinality of any of its bags minus one, i.e. $\text{width}(T) = \max_{i \in \{1,\ldots,n\}}(|B_i| - 1)$. The *treewidth* of a graph G is the minimum width of any of its tree decompositions, whereas the *pathwidth* of G is the minimum width of any of its path decompositions.

3. TIGHT DENSITY RESULTS FOR SMALL VALUES OF k

In this section, we establish the density of 2-layer k-planar graphs for small values of k. We make some preliminary observations regarding the structure of such graphs. A useful concept in this analysis, especially in the case $k = 2$, will be the following:

Definition 3.1. Let G be a topological 2-layer k-planar graph and let $G[i, j|x, y]$, with $1 \leq i \leq j \leq p$ and $1 \leq x \leq y \leq q$, be the topological subgraph of G induced by vertices $\{u_i, \ldots, u_j, v_x, \ldots, v_y\}$. $G[i, j|x, y]$ is a *brick* if it contains exactly two crossing-free edges, namely (u_i, v_x) and (u_j, v_y), which are also crossing-free in G.

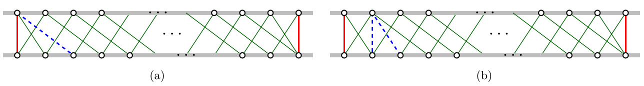

Figure 1. (a) A maximal topological 2-layer 2-planar graph that is not optimal, as shown by the graph in (b). Differences between the two graphs are dashed blue.

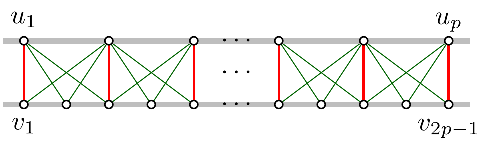

Figure 2. A family of 2-planar graphs on n vertices with $\frac{5}{3}n - \frac{7}{3}$ edges.

The smallest brick, called *trivial*, contains one vertex of one partition set, say $u_i = u_j$, and two consecutive vertices of the second one, say v_x and $v_y = v_{x+1}$.

Observation 3.1. Every optimal topological 2-layer k-planar graph contains crossing-free edges (u_1, v_1) and (u_p, v_q), and hence at least one brick.

Regarding the connectivity we observe the following. If a topological 2-layer k-planar graph G is not connected, we can draw the connected components as consecutive bricks and connect two consecutive bricks with another edge. Hence, we conclude the following:

Observation 3.2. Every optimal topological 2-layer k-planar graph is connected.

Before discussing bounds on the density of 2-layer k-planar graphs, we make an observation about *maximal* topological 2-layer k-planar graphs, that is, in which no edge can be inserted without violating k-planarity or bipartiteness.

Observation 3.3. There exists a maximal topological 2-layer k-planar graph with n vertices and $m = n + O(k)$ edges that is not optimal.

To see this, consider two vertices connected by $k + 1$ vertex-disjoint paths of the same length; see Fig. 1.

3.1 2-Layer 2-Planar Graphs

We begin by providing a lower bound for the density of 2-layer 2-planar graphs:

Lemma 3.1. For every $\beta \geq 0$, there is a 2-layer 2-planar graph with $n = 3\beta + 2$ vertices and $\frac{5}{3}n - \frac{7}{3}$ edges.

Proof. For $\beta = 0$, K_2 is a 2-layer 2-planar graph. For $\beta \geq 1$, we describe a family of graphs where $q = 2p - 1$; see Fig. 2. Each topological graph G of the family consists of a sequence $(b_1, \ldots, b_\beta)$ of $K_{2,3}$-bricks such that b_i and b_{i+1} share a crossing-free edge for $1 \leq i \leq \beta - 1$. Namely, brick b_i consists of vertices $u_i, u_{i+1}, v_{2i-1}, v_{2i}, v_{2i+1}$. Then G has $n = 3\beta + 2$ vertices and $m = 5\beta + 1 = \frac{5}{3}n - \frac{7}{3}$ edges. ■

In fact, it turns out that the lower bound of Theorem 3.1 is tight:

Theorem 3.1. Any 2-layer 2-planar graph on $n \geq 2$ vertices has at most $\frac{5}{3}n - \frac{7}{3}$ edges, and this bound is tight.

Proof. The lower bound follows immediately from Theorem 3.1. We show the upper bound by induction on the number of edges with exactly two crossings. In the base of induction, the graph is in fact 2-layer 1-planar and has at most $\frac{3}{2}n - 2$ edges [34, 37] which

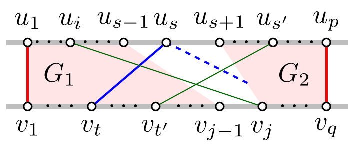

Figure 3. Illustration for the proof of Theorem 3.1.

is at most $\frac{5}{3}n - \frac{7}{3}$ for $n \geq 2$. So assume now that the claim holds for k edges with exactly two crossings.

Next consider a graph with $k + 1$ edges with exactly two crossings and let $e = (u_i, v_j)$ be such an edge that is crossed by edges $e_1 = (u_s, v_t)$ and $e_2 = (u_{s'}, v_{t'})$. Assume w.l.o.g. that $u_s \neq u_{s'}$ (otherwise, due to simplicity, $v_t \neq v_{t'}$) and that $i < s < s'$; see Fig. 3. Now u_s is located in a region delimited by e and e_2. Since e is crossed twice and e_2 is crossed by e, we conclude that u_s has degree at most 2. By removing u_s, its at most two incident edges and e and e_2 we obtain two disconnected subgraphs, namely G_1 induced by vertices $u_1, \ldots, u_{s-1}$ and $v_1, \ldots, v_{j-1}$ and G_2 induced by vertices $u_{s+1}, \ldots, u_p$ and $v_j, \ldots, v_q$.

By induction hypothesis, the n_1-vertex graph G_1 has $m_1 \leq \frac{5}{3}n_1 - \frac{7}{3}$ edges, while the n_2-vertex graph G_2 has $m_2 \leq \frac{5}{3}n_2 - \frac{7}{3}$ edges. Since G consists of G_1, G_2, vertex u_s and at most four additional edges, it has $m \leq m_1 + m_2 + 4 \leq \frac{5}{3}n_1 - \frac{7}{3} + \frac{5}{3}n_2 - \frac{7}{3} + 4 \leq \frac{5}{3}(n-1) - \frac{2}{3} = \frac{5}{3}n - \frac{7}{3}$ edges. This concludes the proof. ■

In the following text, we strengthen our lower bound by characterizing the structure of *optimal* 2-layer 2-planar graphs. We start by excluding two trivial configurations in the following lemma.

Lemma 3.2. Let G be an optimal topological 2-layer 2-planar graph. Then G contains no vertex of degree 1 and no trivial brick.

Proof. Assume that G contains a degree-1 vertex v and consider the graph G' obtained from G by removing v. This graph has $m' = m-1$ edges and $n' = n-1$ vertices. By Theorem 3.1 $m' = \frac{5}{3}n - \frac{7}{3} - 1 = \frac{5}{3}n' - \frac{7}{3} + (\frac{5}{3} - 1) > \frac{5}{3}n' - \frac{7}{3}$; a contradiction to the 2-planarity of G' by Theorem 3.1.

Second, assume that G contains a trivial brick $G[i, i|x, x + 1]$. Then, consider the graph G' obtained from G by identifying vertices v_x and v_{x+1}. Clearly G' has $m' = m - 1$ edges (edges (u_i, v_x) and (u_i, v_{x+1}) coincide in G') and $n' = n - 1$ vertices. This leads to the same contradiction as in the previous case. ■

The next lemma provides useful properties on the structure of bricks in optimal topological 2-layer 2-planar graphs.

Lemma 3.3. Let G be an optimal topological 2-layer 2-planar graph and let $G[i, j|x, y]$ be a brick of G. Then, $j \geq i + 1$ and $y = x + 1$, or $j = i + 1$ and $y \geq x + 1$.

Proof. By Lemma 3.2, $G[i, j|x, y]$ is not a trivial brick. We first observe that u_i is connected to some $v_t \neq v_x$, while v_x is connected to some $u_s \neq u_i$, where both u_s and v_t belong to $G[i, j|x, y]$. If this were not the case, say if u_i were only incident to v_x, then a crossing-free edge (u_{i+1}, v_x), which is not part of G as $G[i, j|x, y]$ contains only (u_i, v_x) and (u_j, v_y) as crossing-free edges, could be inserted, contradicting the optimality of G; see Fig. 4(a).

Assume, for a contradiction, that both $y > x + 1$ and $j > i + 1$. So in the following text assume that (u_i, v_t) and (u_s, v_x) belong to $G[i, j|x, y]$, with $v_t \neq v_x$ and $u_s \neq u_i$, such that there exists no edge $(u_i, v_{t'})$ with $t' > t$ and no edge $(u_{s'}, v_x)$ with $s' > s$.

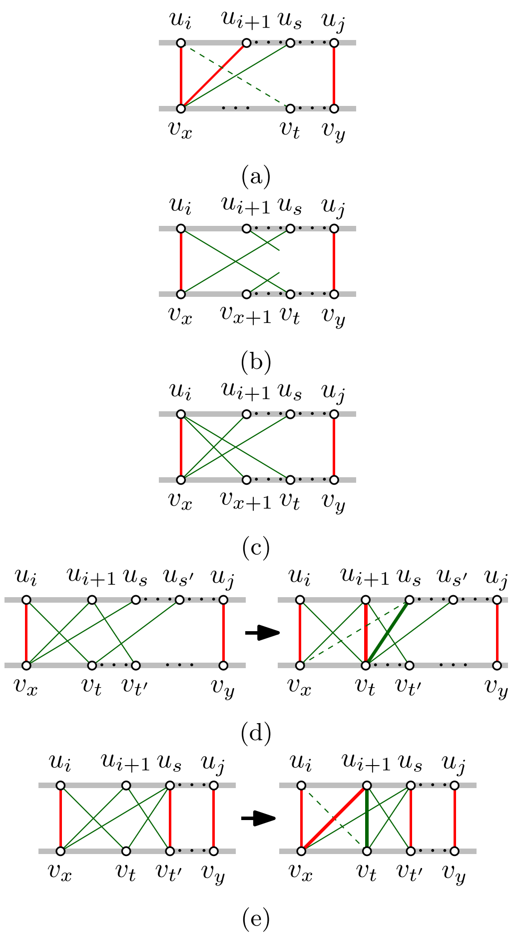

Figure 4. Illustrations for the proof of Lemma 3.3.

Next, we consider u_{i+1} and v_{x+1}. Assume first that $u_{i+1} \neq u_s$ and that $v_{x+1} \neq v_t$. Then, all edges incident to u_{i+1} and v_{x+1} have a crossing with (u_i, v_t) or (u_s, v_x). Since (u_i, v_t) and (u_s, v_x) cross each other, there can be at most two such edges, and thus at least one of u_{i+1} and v_{x+1} has degree at most 1; see Fig. 5(b) and 5(c). This contradicts Lemma 3.2. Hence, assume w.l.o.g. that $v_{x+1} = v_t$. Note that $u_s \neq u_{i+1}$, as otherwise the crossing-free edge (u_{i+1}, v_{x+1}), which is not present in G as $G[i, j|x, y]$ is a brick, could be inserted, contradicting the optimality of G. In addition, $u_s = u_{i+2}$, since otherwise u_{i+1} and u_{i+2} could only be incident to a total of two edges, by the same argument as above, resulting in a degree-1 vertex, which contradicts Lemma 3.2. Finally, by the same kind of argument, (u_{i+1}, v_x) is necessarily part of G as otherwise u_{i+1} would be a degree-1 vertex.

By Lemma 3.2, both u_{i+1} and v_{x+1} have degree at least 2. Let $u_{s'}$ and $v_{t'}$ denote the neighbors of v_{x+1} and u_{i+1}, respectively, such that s' and t' are maximal. First assume that $t' \neq t$. If $s' = i+1$, the crossing-free edge (u_s, v_t) can be inserted as edge (u_s, v_x) is crossed by edges (u_i, v_t) and $(u_{s'}, v_t)$, i.e. there can be no other edge crossing (u_s, v_x). This contradicts the optimality of G. We observe that edge $(u_{i+1}, v_{t'})$ is crossed by edges (u_s, v_x) and $(u_{s'}, v_t)$. If $u_s \neq u'_s$, we can obtain a topological 2-layer 2-planar graph G' by removing edge (u_s, v_x) and inserting edges (u_s, v_t) and (u_{i+1}, v_t); see Fig. 4(d). Note that both newly inserted edges cannot be part of G by 2-planarity, hence this clearly contradicts the optimality of G. If $u_s = u_{s'}$, we can obtain a topological 2-layer 2-planar graph G' by removing edge (u_i, v_t) and inserting the edge (u_{i+1}, v_t); see Fig. 4(e). This again contradicts the optimality of G as G' has the same number of edges

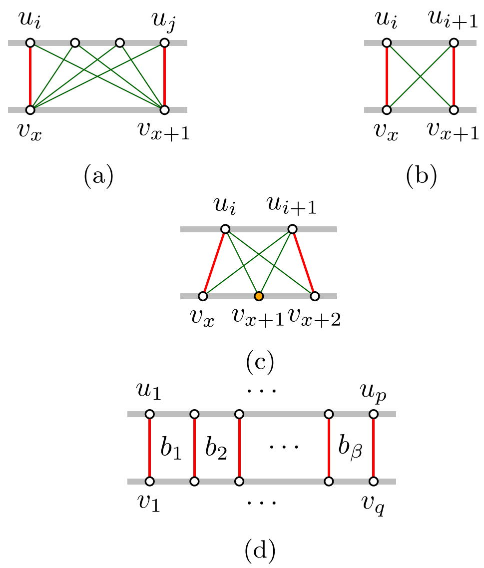

Figure 5. The unique 2-layer drawings of (a) $K_{2,4}$; (b) $K_{2,2}$; (c) $K_{2,3}$. (d) An optimal 2-layer 2-planar graph is a sequence of bricks joint at crossing-free edges.

as G and should thus be optimal, however $G'[i, i+1|x, x]$ is a trivial brick which is forbidden by Lemma 3.2. We conclude that $t' = t$.

Since (u_s, v_x) is crossed by edges (u_i, v_t) and (u_{i+1}, v_t), we conclude that (u_s, v_t) can be inserted without crossings, contradicting the optimality of G. ■

By Lemma 3.2 and 3.3, we get that every brick must be a $K_{2,h}$ for some $h \geq 2$. The following observation shows that $h \leq 3$; see also Fig. 5(a):

Observation 3.4. The complete bipartite graph $K_{2,4}$ is not 2-layer 2-planar.

We are now ready to characterize the optimal 2-layer 2-planar graphs.

Theorem 3.2. A 2-layer 2-planar graph is optimal if and only if it consists of a sequence of $K_{2,3}$s such that consecutive $K_{2,3}$s share one crossing-free edge.

Proof. Lemmas 3.2 and 3.3, and Observation 3.4 imply that G contains only $K_{2,2}$- and $K_{2,3}$-bricks; see Fig. 5(b), and 5(c). Moreover, the crossing-free edges separate G into a sequence of β bricks $(b_1, \ldots, b_\beta)$ such that b_i and b_{i+1} share one crossing-free edge. Let β_2 denote the number of $K_{2,2}$-bricks. Then, G has $\beta - \beta_2$ $K_{2,3}$-bricks. Moreover, $n = 2\beta + 2 + (\beta - \beta_2) = 3\beta - \beta_2 + 2$ since each of the $\beta + 1$ crossing-free edges is incident to two distinct vertices, while each $K_{2,3}$-brick contains an additional vertex; see Fig. 5(c). Finally, $m = \beta + 1 + 2\beta_2 + 4(\beta - \beta_2) = 5\beta - 2\beta_2 + 1$ since every $K_{2,2}$-brick contains two crossing edges, while every $K_{2,3}$-brick contains four. For a fixed value of n, $\beta = \frac{1}{3}n + \frac{1}{3}\beta_2 - \frac{2}{3}$ and the density is $m = \frac{5}{3}n - \frac{1}{3}\beta_2 - \frac{7}{3}$. This is clearly maximized for $\beta_2 = 0$. Hence, the maximum density is $m = \frac{5}{3}n - \frac{7}{3}$, which is tightly achieved for graphs in which every brick is a $K_{2,3}$. ■

3.2 2-Layer 3-Planar Graphs

Next, we give a tight bound on the density of 2-layer 3-planar graphs. We first present a lower bound construction:

Lemma 3.4. For every $n \geq 3$, there is a 2-layer 3-planar graph with n vertices and $2n - 4$ edges.

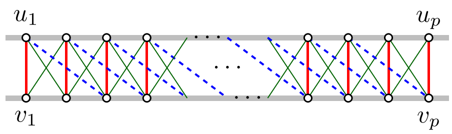

Figure 6. A family of 3-planar graphs on n vertices with $2n-4$ edges.

Proof. We first consider the case where n is even and describe a family of graphs where $p = q = n/2$; refer to Fig. 6. Each graph has the following edges:

- (u_i, v_i) for $1 \le i \le p$ (bold red edges in Fig. 6),
- (u_i, v_{i+1}) for $1 \le i \le p-1$ and (u_i, v_{i-1}) for $2 \le i \le p$ (green edges in Fig. 6) and
- (u_i, v_{i+2}) for $1 \le i \le p-2$ (dashed blue edges in Fig. 6).

Vertices u_1, u_{p-1}, v_2 and v_p have degree 3, u_p and v_1 have degree 2 and all other vertices have degree 4, yielding $4n-8$ for the sum of the vertex degrees and hence $2n-4$ edges.

For odd values of n, it is enough to remove vertex v_1 from the construction for $n+1$, which has degree 2, to obtain the claimed result. ∎

Before we proceed to prove a corresponding upper bound, we show the following lemma which will serve as a key ingredient.

Lemma 3.5. Let (u_i, v_y), (u_s, v_t) and (u_j, v_x) be a triple of pairwise crossing edges in a topological 2-layer 3-planar graph such that $1 \le i < s < j \le p$ and $1 \le x < t < y \le q$. Then the number of edges incident to u_s or v_t is at most 3, i.e. $d(u_s) + d(v_t) - 1 \le 3$.

Proof. Consider such a triple of edges. If u_s is connected to another vertex $v \ne v_t$, edge (u_s, v) crosses one of (u_i, v_y) and (u_j, v_x). The same is true, if v_t is connected to another vertex $u \ne u_s$. Since (u_i, v_y) and (u_j, v_x) both have two crossings from the triple of crossing edges, u_s and v_t can only be incident to a total of two edges which are not (u_s, v_t). ∎

We are now ready to provide the corresponding density upper bound:

Theorem 3.3. Let G be a topological 2-layer 3-planar graph on $n \ge 3$ vertices. Then G has at most $2n-4$ edges, which is a tight bound. Moreover, if G is optimal, it is quasiplanar.

Proof. We will show the statements by proving a stronger property, namely, that G has at most $2n-4-k$ edges, where k is the number of triples of edges that pairwise cross. The proof is by induction on k. If $k = 0$, G is quasiplanar and has at most $2n-4 = 2n-4-0$ edges by Theorem 2.1. So assume that any 2-layer 3-planar graph with k triples of edges that pairwise cross has at most $2n-4-k$ edges.

Now let G be a 2-layer 3-planar graph with $k+1$ triples of edges that pairwise cross and let (u_i, v_y), (u_s, v_t) and (u_j, v_x) be a triple of pairwise crossing edges in G with $1 \le i < s < j \le p$ and $1 \le x < t < y \le q$. By Lemma 3.5, vertices u_s and v_t are incident to a total of at most three edges. By induction hypothesis, $G \setminus \{u_s, v_t\}$ has at most $2(n-2)-4-k$ edges. Hence, G has $m \le 2(n-2)-k-4+3 = 2n-k-5 = 2n-4-(k+1)$ edges.

By Theorem 3.4, there is a 2-layer 3-planar graph with n vertices and $2n-4$ edges for every $n \ge 3$. Since we have shown that $2n-4-k$ is an upper bound for the number of edges, we conclude that G contains no triple of pairwise crossing edges if it is optimal. Thus, if G is optimal, it is quasiplanar and has at most $2n-4$ edges for $n \ge 3$, by Theorem 2.1. ∎

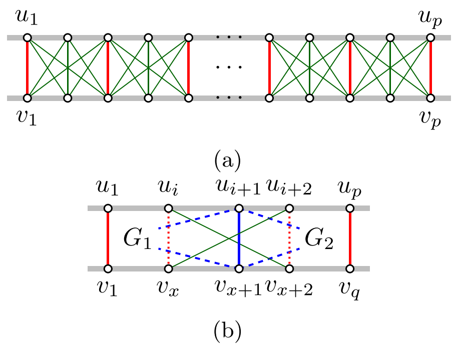

Figure 7. (a) A family of 4-planar graphs on n vertices with $2n-3$ edges. (b) A triple of pairwise crossing edges and at most 4 additional edges separates an optimal 2-layer 4-planar graph into graphs G_1 and G_2.

3.3 2-Layer 4-Planar Graphs

We first present a lower bound construction for this class of graphs:

Lemma 3.6. For every $\beta \ge 0$, there is a 2-layer 4-planar graph with $n = 4\beta + 2$ vertices and $2n-3$ edges.

Proof. For $\beta = 0$, K_2 is a 2-layer 3-planar graph. For $\beta \ge 1$, we describe a family of graphs where $p = q = n/2$; see Fig. 7(a). Each topological graph G of the family consists of a sequence $(b_1, \dots, b_\beta)$ of $K_{3,3}$-bricks such that b_i and b_{i+1} share a crossing-free edge for $1 \le i \le \beta - 1$. Then G has $n = 4\beta + 2$ vertices and $m = 8\beta + 1 = 2n-3$ edges. ∎

Next, we provide a matching upper bound and prove that the graph family from the proof of Theorem 3.6 is identical to the family of optimal 2-layer 4-planar graphs.

Theorem 3.4. Any 2-layer 4-planar graph on $n \ge 2$ vertices has at most $2n-3$ edges, and this bound is tight. Further, a 2-layer 4-planar graph is optimal if and only if it is either a single edge or consists of a sequence of $K_{3,3}$s such that consecutive $K_{3,3}$s share one crossing-free edge.

Proof. The lower bound follows immediately from Theorem 3.6. We will show the upper bound and the characterization of optimal 2-layer 4-planar graphs by induction on the number k of triples of edges that pairwise cross. In the base case, $k = 0$ and G is quasiplanar and thus by Theorem 2.1 it has at most $2n-4$ edges. Moreover, in the degenerate case of $n = 2$, it has at most $1 = 2n-3$ edges. Hence, the statement follows in the base case.

Next, assume that $k > 0$ and let (u_i, v_y), (u_s, v_t) and (u_j, v_x) for some $1 \le i < s < j \le p$ and $1 \le x < t < y \le q$ be triple of pairwise crossing edges.

First, assume that that there is a vertex $u_{s'}$ such that $i < s' < j$ and $s' \ne s$. Each of the edges (u_i, v_y) and (u_j, v_x) has two crossings from the triple of pairwise crossing edges, so each of them can only be crossed by two more edges. Hence, there are at most five edges incident to the vertices $u_s, u_{s'}, v_t$ (including the edge (u_s, v_t)). Then the graph G' obtained by removing vertices $u_s, u_{s'}, v_t$ has $n' =$

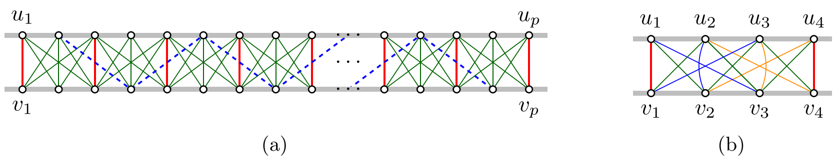

Figure 8. (a) A family of 5-planar graphs on n vertices with $\frac{9}{4}n - \frac{9}{2}$ edges. (b) Graph $\mathcal{S}$ with $n = 8$ vertices and $m = 14 > \frac{9}{4} \cdot 8 - \frac{9}{2} = 13.5$ edges.

$n-3$ vertices and $k' < k$ triples of pairwise crossing edges. Thus, by induction hypothesis, it has $m' \leq 2n' - 3$ edges. Then $m \leq m' + 5 \leq 2n' - 3 + 5 = 2(n-3) + 2 = 2n - 4$. Note that in this case, G cannot be optimal. The case where there exists a vertex $v_{t'}$ such that $x < t' < y$ and $t' \neq t$ is symmetric.

It remains to consider the case where no such vertex $u_{s'}$ and $v_{t'}$ exists, that is, $s = i+1, j = i+2, t = x+1$ and $y = x+2$. Consider the subgraph G_1 induced by $u_1, \ldots, u_i$ and $v_1, \ldots v_x$ and the subgraph G_2 induced by $u_{i+2}, \ldots, u_p$ and $v_{x+2}, \ldots v_q$.

Assume first that there is an edge $(u_h, v_z) \neq (u_i, v_{x+2})$ such that $1 \leq h \leq i$ and $x+2 \leq z \leq q$. This edge crosses (u_s, v_t) and (u_{i+2}, v_x). Then, (u_h, v_z), (u_{i+2}, v_x) and (u_s, v_t) form a triple of pairwise crossing edges where u_i and u_{i+1} are between u_h and u_{i+2} (if $h < i$) or v_{x+1} and v_{x+2} are between v_x and v_z (if $z > x+2$). Then, it is possible to apply the argument of the previous case on the new triple. The case where there is an edge $(u_h, v_z) \neq (u_{i+2}, v_x)$ such that $i+2 \leq h \leq p$ and $1 \leq z \leq x$ is symmetric.

Second, assume the case where G_1 is connected to G_2 only by the edges (u_i, v_{x+2}), (u_{i+2}, v_x) and paths traversing u_s or v_t; see Fig. 7(b). Clearly, each of G_1 and G_2 has less triples of pairwise crossing edges than G, so the induction hypothesis applies for them. As mentioned before, since (u_i, v_{x+2}) and (u_{i+2}, v_x) have two crossings from the triple of pairwise crossing edges, u_{i+1} and v_{x+1} can only be incident to a total of five edges including edge (u_{i+1}, v_{x+1}), that is, G contains at most seven edges that are neither part of G_1 nor of G_2. Hence, G can be split into (possibly optimal) subgraphs G_1 and G_2 and two isolated vertices u_{i+1} and v_{x+1} by removing seven edges. Let n_1 and n_2 be the number of vertices of G_1 and G_2, respectively, where $n = n_1 + n_2 + 2$. By induction hypothesis, G_1 and G_2 have $m_1 \leq 2n_1 - 3$ and $m_2 \leq 2n_2 - 3$ edges, respectively. Then, $m \leq m_1 + m_2 + 7 \leq 2(n_1+n_2) - 6 + 7 = 2(n_1+n_2) + 1 = 2(n-2) + 1 = 2n - 3$.

If G is optimal, the smaller equal relations in the previous equation must be all tight. In particular, G_1 and G_2 must be optimal. Recall also that u_{i+1} and v_{x+1} are incident to four edges with endpoints in G_1 or G_2 as stated above. By induction hypothesis each of G_1 and G_2 is either a single edge or a sequence of $K_{3,3}$'s such that consecutive $K_{3,3}$'s share one crossing-free edge (for the latter case refer also to Fig. 7(a) for an illustration). In either case, the requirement that u_{i+1} and v_{x+1} are incident to four edges with endpoints in G_1 or G_2 implies that u_{i+1} must be adjacent to v_x in G_1 and to v_{x+2} in G_2, while v_{x+1} must be adjacent to u_i in G_1 and to u_{i+2} in G_2. Namely, if G_1 is a single edge, it only consists of u_i and v_x, while if it is a sequence of $K_{3,3}$'s, the edges (u_i, v_{x-2}) and (u_{i-2}, v_x) are already crossed four times within G_1. A symmetric argument applies for G_2. As a result, vertices $u_i, u_{i+1}, u_{i+2}, v_x, v_{x+1}, v_{x+2}$ induce a $K_{3,3}$ that shares the crossing-free edge (u_i, v_x) with the last $K_{3,3}$ of G_1 and the crossing-free edge (u_{i+2}, v_{x+2}) with the first $K_{3,3}$ of G_2. We conclude that G is a sequence of $K_{3,3}$'s such that consecutive $K_{3,3}$'s share one crossing-free edge. ■

3.4 2-Layer 5-Planar Graphs

We first provide a lower bound construction for this class of graphs:

Lemma 3.7. For every $\beta \geq 2$, there is a 2-layer 5-planar graph with $n = 4\beta + 2$ vertices and $\frac{9}{4}n - \frac{9}{2}$ edges. Further, there exists a topological 2-layer 5-planar graph $\mathcal{S}$ with $n_{\mathcal{S}} = 8$ vertices and $m_{\mathcal{S}} = 14 > \frac{9}{4}n_{\mathcal{S}} - \frac{9}{2}$ edges.

Proof. For the first part of the statement, we augment the construction from Theorem 3.6 by a path of length $\beta - 1$; see the dashed blue edges in Fig. 8(a). Note that $\beta \geq 2$ implies that there exists at least one edge in this path. The obtained graph has $n = 4\beta + 2$ vertices and $m = 9\beta = \frac{9}{4}n - \frac{9}{2}$ edges. This completes the first part of the statement.

The second part of the statement shows that for the specific value $n = 8$, we can provide a denser lower bound construction. The corresponding graph can be found in Fig. 8(b). ■

We show that the graph $\mathcal{S}$ is in fact an exception, by demonstrating that the lower bound construction in Theorem 3.7 is tight for all other values of n.

Theorem 3.5. Any 2-layer 5-planar graph on $n \geq 3$ vertices has at most $\frac{9}{4}n - \frac{9}{2}$ edges, except for the graph $\mathcal{S}$ shown in Fig. 8(b) which has eight vertices and 14 edges. Further, the upper bound is tight.

Proof. The lower bound follows immediately from Theorem 3.7. If $G = \mathcal{S}$, the upper bound stated in the theorem trivially holds. Otherwise, we show the statement by induction on the number k of triples of edges that pairwise cross. In the base case $k = 0$ and by Theorem 2.1 graph G has at most $2n-4$ edges, which is smaller than $\frac{9}{4}n - \frac{9}{2}$ for $n \geq 3$.

In the inductive case $k > 0$, consider a triple of pairwise crossing edges (u_i, v_y), (u_s, v_t) and (u_j, v_x) for some $1 \leq i < s < j \leq p$ and $1 \leq x < t < y \leq q$. Let G_1 be the subgraph induced by vertices $u_1, \ldots, u_i$ and $v_1, \ldots v_x$ and G_2 be the subgraph induced by vertices $u_j, \ldots, u_p$ and $v_y, \ldots v_q$. We consider two cases depending on the presence of a vertex $w \notin \{u_s, v_t\}$ that belongs neither to G_1 nor to G_2.

Case 1: w **exists.** Since every incidence to u_s, v_t and w implies a crossing on (u_i, v_y) or (u_j, v_x), one of u_s, v_t and w has degree at most two. Let w^* denote this vertex. First, assume that the graph G' obtained by removing w^* is isomorphic to $\mathcal{S}$. We observe that if we insert a vertex w^* between u_1 and u_4 into $\mathcal{S}$, it would be located in a region bounded by edges that are crossed five times by edges not incident to w^*. Hence, it would be an isolated vertex. Then, $n' = n - 1$ and $m' = \frac{9}{4}n' - 4 = m$. It follows that $m = \frac{9}{4}(n-1) - 4 = \frac{9}{4}n - \frac{12.5}{2}$. Second, assume that $G' \neq \mathcal{S}$, i.e. it has $n' = n - 1$ vertices and $m' \leq \frac{9}{4}n' - \frac{9}{2}$ edges by induction. Also, $m \leq m' + 2$ since w^* has degree at most 2. Then, $m \leq \frac{9}{4}(n-1) - \frac{9}{2} + 2 = \frac{9}{4}n - \frac{9.5}{2}$.

Case 2: w **does not exist.** In this case u_s is the only vertex between u_i and u_j and v_t is the only vertex between v_x and v_y. We first observe that there is no edge (u_h, v_z) different from (u_i, v_y) and (u_j, v_x) that directly connects G_1 with G_2. Namely, suppose that there is such an edge (u_h, v_z) with $h < i$ and $z \geq y$ (the other cases are symmetric); see Fig. 9(a). Then, (u_h, v_z) creates a triple of

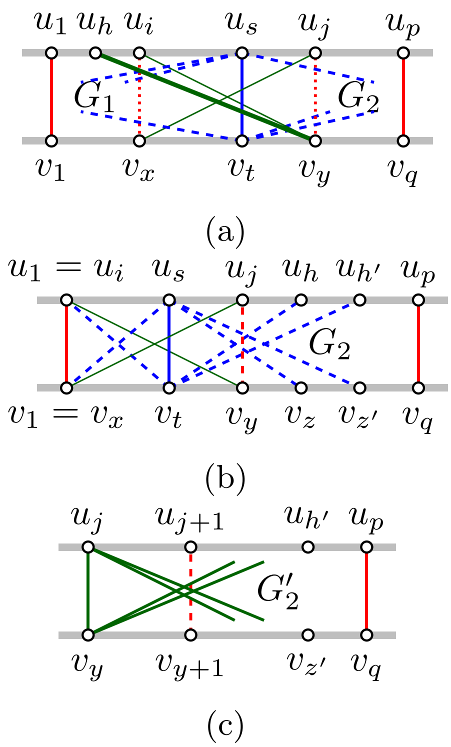

Figure 9. (a) A triple (u_i, v_y), (u_s, v_t), (u_j, v_x) of pairwise crossing edges and at most six other edges separates an optimal 2-layer 5-planar graph into subgraphs G_1 and G_2. If G_1 consists of a single edge, (b) there can be edges (u_s, v_z), $(u_s, v_{z'})$, (u_h, v_t), $(u_{h'}, v_t)$, in which case (c) G_2 consists of a graph G'_2, vertices u_j, v_y and at most four of the green edges.

pairwise crossing edges with (u_s, v_t) and (u_j, v_x), where u_i is located between u_h and u_j. Then Case 1 applies for this triple of edges.

Since (u_i, v_y) and (u_j, v_x) have two crossings from the triple of pairwise crossing edges, the vertices u_s and v_t are incident to a total of at most seven edges including the edge (u_s, v_t). Hence, G can be split into subgraphs G_1 and G_2 and two isolated vertices u_s and v_t by removing nine edges. Since (u_i, v_y), (u_j, v_x) and (u_s, v_t) is a triple of pairwise crossing edges that does not belong to G_1 or G_2, both G_1 and G_2 have fewer triples of pairwise crossing edges than G.

Let n_1 and n_2 denote the number of vertices of G_1 and G_2, respectively. Clearly, $n = n_1 + n_2 + 2$. We consider the following subcases.

Case 2.1: G_1 or G_2 is isomorphic to $\mathcal{S}$. We assume that G_1 is isomorphic to $\mathcal{S}$; the case where G_2 is isomorphic to $\mathcal{S}$ is symmetric. We observe that $(u_4, v_4) = (u_i, v_x)$ is a crossing-free edge of G, since edges (u_2, v_4) and (u_4, v_2) have five crossings each within $\mathcal{S}$; see Fig. 8(b). Then, consider the graph G' obtained from G by the removal of vertices u_1, u_2, u_3, v_1, v_2 and v_3. Here, we consider two cases. If G' is also isomorphic to $\mathcal{S}$, then G contains $n = 14$ vertices and $m = 27 = \frac{9}{4} \cdot 14 - \frac{9}{2}$ edges. Otherwise, G' has $n' = n - 6$ vertices and $m' \le \frac{9}{4}n' - \frac{9}{2}$ edges by induction. Then, $m = m' + 13 \le \frac{9}{4}(n-6) - \frac{9}{2} + 13 = \frac{9}{4}n - \frac{10}{2}$.

Case 2.2: Neither G_1 nor G_2 is isomorphic to $\mathcal{S}$ and $n_1, n_2 \ge 3$. By induction, G_1 and G_2 have at most $\frac{9}{4}n_1 - \frac{9}{2}$ and $\frac{9}{4}n_2 - \frac{9}{2}$ edges, respectively. We conclude that $m \le \frac{9}{4}(n_1 + n_2) - 2 \cdot \frac{9}{2} + 9 \le \frac{9}{4}(n_1 + n_2) - 9 + 9 = \frac{9}{4}(n_1 + n_2) = \frac{9}{4}(n-2) = \frac{9}{4}n - \frac{9}{2}$.

Case 2.3: $n_1 = 2$ or $n_2 = 2$. Here, we assume that $n_1 = 2$, i.e. G_1 coincides with edge (u_1, v_1); see Fig. 9(b). The case where $n_2 = 2$ is symmetric. If either u_1 or v_1 has degree at most two, consider the graph G' obtained by removing this vertex. Then G' has $m' \le \frac{9}{4}(n-1) - \frac{9}{2}$ edges. Conversely, $m \le m' + 2 \le \frac{9}{4}(n-1) - \frac{9}{2} + 2 = \frac{9}{4}n - \frac{9.5}{2}$. So assume now that both u_1 and v_1 have degree at least three.

Because u_1 belongs to G_1, u_1 can only be incident to $v_x = v_1$, $v_t = v_2$ and $v_y = v_3$. Hence, edge (u_1, v_2) must exist since otherwise u_1 would have degree two. Symmetrically, edge (u_2, v_1) is also present. Then, u_2 and v_2 can only be incident to a total of seven edges, if there are edges (u_2, v_z), $(u_2, v_{z'})$, (u_h, v_t) and $(u_{h'}, v_t)$ for some $3 \le h < h' \le p$ and $3 \le z < z' \le q$. If u_2 and v_2 were only incident to at most six edges in total, G would have $m \le m_2 + 9$ edges (as it contains all edges except for those that are incident to G_1 or u_2 or v_2) and $n_2 = n - 4$ vertices. Since by induction $m_2 \le \frac{9}{4}n_2 - \frac{9}{2}$, we can conclude that G has at most $m \le m_2 + 9 \le \frac{9}{4}(n-4) + \frac{9}{2} = \frac{9}{4}n - \frac{9}{2}$ edges. Therefore, we assume in the following that $n_2 \ge 4$.

If $n_2 = 4$, we observe that $j = h = 3$, $h' = p = 4$, $y = z = 3$ and $z' = q = 4$. Then, G is a proper subgraph of $\mathcal{S}$ (as $\mathcal{S}$ is a $K_{4,4}$ without edges (u_1, v_4) and (u_4, v_1)) and has less than $\frac{9}{4}n - \frac{9}{2}$ edges (since $\mathcal{S}$ has $\frac{9}{4}n - \frac{9}{2} + \frac{1}{2}$ edges).

It remains to consider the case $n_2 > 4$. We observe that all edges in G_2 that are incident to u_j will cross edge $(u_{h'}, v_2)$. Since $(u_{h'}, v_2)$ already has three crossings, it follows that the degree of u_3 in G_2 is at most two. Symmetrically, the degree of v_3 in G_2 is at most two. Consider the graph G'_2 obtained from G_2 by removing u_3 and v_3; see Fig. 9(c). Since $n_2 > 4$, G'_2 has $n'_2 \ge 3$ vertices. First assume that G'_2 is isomorphic to $\mathcal{S}$. Then, edge (u_4, v_4) is planar, and u_3 and v_3 can only be incident to (u_3, v_3), (u_3, v_4) and (u_4, v_3). Then, G_2 has $n_2 = 10$ vertices and $m_2 \le 17 = \frac{9}{4}n_2 - \frac{11}{2}$ edges.

Next, assume that G'_2 is not isomorphic to $\mathcal{S}$. We consider two cases. If (u_3, v_3) is not in G_2 consider the graph G^*_2 obtained from G_2 by inserting edge (u_3, v_3). Clearly, G^*_2 is 2-layer 5-planar and hence has at most $m^*_2 \le \frac{9}{4}n_2 - \frac{9}{2}$ edges. Since $m_2 = m^*_2 - 1$, it follows that $m_2 \le \frac{9}{4}n_2 - \frac{11}{2}$. So assume that (u_3, v_3) is part of G_2. Then, there are at most three edges in G_2 that are not in G'_2 since $(u_2, v_{z'})$ is crossed by (u_3, v_1), (u_h, v_2), $(u_{h'}, v_2)$ and (u_3, v_3), while $(u_{h'}, v_2)$ is crossed by (u_1, v_3), (u_2, v_z), $(u_2, v_{z'})$ and (u_3, v_3). By induction hypothesis G'_2 has $m'_2 \le \frac{9}{4}n'_2 - \frac{9}{2}$ edges. Then, G_2 has $m_2 \le m'_2 + 3 \le \frac{9}{4}(n_2 - 2) - \frac{9}{2} + 3 = \frac{9}{4}n_2 - \frac{12}{2}$ edges.

We conclude that $m_2 \le \frac{9}{4}n_2 - \frac{11}{2}$. Thus, $m \le m_1 + m_2 + 9 \le 1 + \frac{9}{4}n_2 - \frac{11}{2} + 9 \le \frac{9}{4}(n-4) - \frac{11}{2} + 10 = \frac{9}{4}(n-4) + \frac{9}{2} = \frac{9}{4}n - 9 + \frac{9}{2} = \frac{9}{4}n - \frac{9}{2}$.

Since the induction holds in all the cases, the statement follows. ■

4. A CROSSING LEMMA AND GENERAL DENSITY BOUNDS

In this section, we generalize the well-known Crossing Lemma [63–65] to a meta Crossing Lemma for general graphs under some restriction $\mathcal{R}$ (e.g. $\mathcal{R}$ can be 'bipartite' or '2-layer') in the presence of density bounds on $\mathcal{R}$-restricted k-planar graphs for small k; see Theorem 4.2. This also yields a density upper bound for $\mathcal{R}$-restricted k-planar graphs for larger values of k. We remark that this approach has already been used to establish density bounds for k-planar and bipartite k-planar graphs [8, 66]; however, to the best of our knowledge it was never formulated as a meta theorem before.

In order to formalize, we denote by $\mathcal{R}$ a restriction on graphs (again, e.g. $\mathcal{R}$ can be 'bipartite' or '2-layer') and call a graph *$\mathcal{R}$-restricted k-planar graph* if it admits a k-planar drawing satisfying the restriction $\mathcal{R}$. We assume that for a fixed $t > 0$, there are $\alpha_i, \beta_i \in \mathbb{R}$ for $i \in \{0, \dots, t-1\}$ such that $m \le \alpha_i n - \beta_i$ is an upper bound for the number of edges in $\mathcal{R}$-restricted i-planar graphs. Let $\alpha := \sum_{i=0}^{t-1} \alpha_i$ and $\beta := \sum_{i=0}^{t-1} \beta_i$. The proofs of the next lemmas follow the probabilistic technique of Chazelle, Sharir and Welzl (see e.g. [[9], Chapter 35]). We first give an auxiliary lemma that will be used in the proof of our meta Crossing Lemma.

Lemma 4.1. Let G be a simple $\mathcal{R}$-restricted graph with $n \geq 4$ vertices and m edges. Then, the following inequality holds for the crossing number $cr(G)$:

$$cr(G) \geq tm - \alpha n + \beta. \tag{1}$$

Proof. Clearly, Inequality (1) holds for $m \leq \alpha_0 n - \beta_0$. Next, assume that $m > \alpha_0 n - \beta_0$. Then there exist at least $m - (\alpha_0 n - \beta_0)$ edges in G that have at least one crossing. If $m > \alpha_1 n - \beta_1$, there exists at least $m - (\alpha_1 n - \beta_1)$ edges in G that have at least two crossings. Iteratively, we obtain that, if $m > \alpha_{i-1} n - \beta_{i-1}$, there exists at least $m - (\alpha_i n - \beta_i)$ edges in G that have at least i crossings. Therefore, we obtain

$$cr(G) \geq \sum_{i=0}^{t-1} [m - (\alpha_i n - \beta_i)] = tm - \alpha n + \beta,$$

which concludes the proof. ■

Next, we prove the meta Crossing Lemma for $\mathcal{R}$-restricted graphs:

Lemma 4.2. crossingLemma Let G be a simple $\mathcal{R}$-restricted graph with $n \geq 4$ vertices and $m \geq \frac{3\alpha}{2t} n$ edges. The following inequality holds for the crossing number $cr(G)$:

$$cr(G) \geq \frac{4t^3}{27\alpha^2} \frac{m^3}{n^2}. \tag{2}$$

Proof. Consider a drawing Γ of G with $cr(G)$ crossings and let $\pi = \frac{3\alpha n}{2tm} \leq 1$. With probability π choose every vertex of G independently and let G_π denote the subgraph of G induced by the chosen vertices, and Γ_π the subdrawing of Γ representing G_π. Consider random variables N_π, M_π and C_π denoting the number of vertices and edges in G_π and the number of crossings in Γ_π, respectively. By Lemma 4.1, it holds that $C_\pi \geq tM_\pi - \alpha N_\pi + \beta$. Taking expectations on this relationship, we obtain

$$\pi^4 cr(G) \geq t\pi^2 m - \alpha\pi n \implies cr(G) \geq \frac{tm}{\pi^2} - \frac{\alpha n}{\pi^3}$$

We obtain Inequality (2) by substituting $\pi = \frac{3\alpha n}{2tm}$ into the inequality above. ■

The meta Crossing Lemma is used to obtain the following theorem regarding the density.

Lemma 4.3. kPlanarUpperbound Let G be a simple $\mathcal{R}$-restricted k-planar graph with $n \geq 4$ vertices for some $k \geq t$. Then

$$m \leq \max\left\{1, \sqrt{\frac{3}{2t}}\sqrt{k}\right\} \cdot \frac{3\alpha}{2t} n.$$

Proof. We follow closely the proof for corresponding statements for k-planar and bipartite k-planar graphs [8, 66]. If $m \leq \frac{3\alpha}{2t} n$, the proof follows immediately. Otherwise, we obtain from Theorem

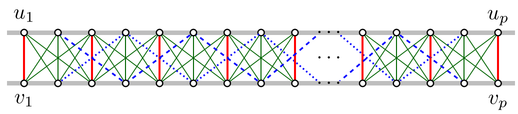

Figure 10. A family of 6-planar graphs on n vertices with $\frac{5}{2}n - 6$ edges.

4.2 and from the assumption that G is k-planar:

$$\frac{4t^3}{27\alpha^2} \frac{m^3}{n^2} \leq cr(G) \leq \frac{1}{2} mk.$$

This implies

$$m \leq \frac{3\alpha}{2t} \sqrt{\frac{3}{2t}} \sqrt{k} n,$$

which completes the proof. ■

We apply Theorems 4.2 and 4.3 to 2-layer k-planar graphs for $t = 6$. By [34], Theorems 3.1 and 3.3 to 3.5, we have $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, \frac{3}{2}, \frac{5}{3}, 2, 2, \frac{9}{4})$, yielding $\alpha = \frac{125}{12}$. By substituting the numbers in Theorem 4.2 we obtain the following.

Theorem 4.1. Let G be a simple 2-layer graph with $n \geq 4$ vertices and $m \geq \frac{125}{48} n$ edges. Then, the following inequality holds for the crossing number $cr(G)$:

$$cr(G) \geq \frac{4.608}{15.625} \frac{m^3}{n^2} \approx 0.295 \frac{m^3}{n^2}.$$

By plugging the result into Theorem 4.3 we obtain:

Theorem 4.2. Let G be a simple 2-layer k-planar graph with $n \geq 4$ vertices for some $k > 5$. Then

$$m \leq \frac{125}{96} \sqrt{k} \cdot n.$$

Note that for 2-layer 6-planar graphs, Theorem 4.2 certifies that $m \leq 3.19n$. We can show that there is only a gap of $0.69n$ toward an optimal solution:

Theorem 4.3. For every $\beta \geq 2$, there is a 2-layer 6-planar graph with $n = 4\beta + 2$ vertices and $\frac{5}{2}n - 6$ edges.

Proof. We augment the construction from Theorem 3.7 by a second path of length $\beta - 1$; refer to the dotted blue path in Fig. 10. The obtained graph has $n = 4\beta + 2$ vertices and $m = 10\beta - 1 = \frac{5}{2}n - 6$ edges. ■

In the next theorem, we additionally show that the multiplicative constant from Theorem 4.2 is within a factor of 1.84 of the optimal achievable upper bound.

Theorem 4.4. For any k, there exist infinitely many 2-layer k-planar graphs with n vertices and $m = \left\lfloor \sqrt{k/2} \right\rfloor n - \mathcal{O}(k) \approx 0.707\sqrt{k}n - \mathcal{O}(k)$ edges.

Proof. We choose $p = q$. Depending on k, we choose a parameter ℓ that we will define later. We connect vertex u_i to at most ℓ vertices $v_{i+1} \ldots, v_{\min\{i+\ell,p\}}$. Similarly, we connect vertex v_i to

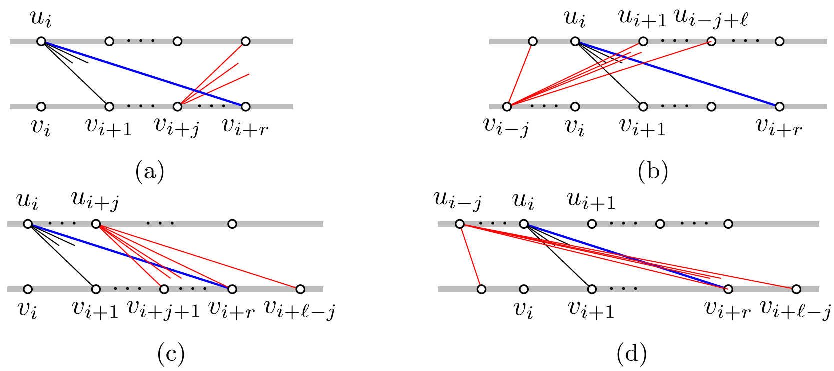

Figure 11. Illustrations for the proof of Theorem 4.4.

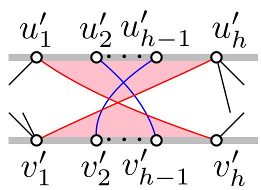

Figure 12. A set of h pairwise crossing edges in a topological 2-layer graph.

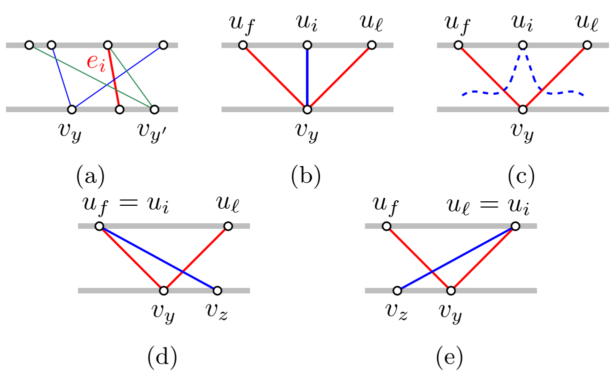

Figure 13. Illustrations for the proof of Theorem 5.2.

vertices $u_{i+1} \dots, u_{\min\{i+\ell,p\}}$. Orient the edges from lower to higher index. Then, each vertex (except for those with indices at least $p-\ell$) has ℓ outgoing edges.

Moreover, edge (u_i, v_{i+r}) (for $1 \le r \le \ell$) is only crossed by

- at most ℓ outgoing edges from vertex v_{i+j} for $0 \le j \le r-1$; see Fig. 11(a),
-
- at most $\ell - j$ outgoing edges from vertex v_{i-j} for $1 \le j \le \ell - 1$ which connect v_{i-j} to vertices $u_{i+1}, \dots, u_{\min\{i+\ell-j,p\}}$; see Fig. 11(b),
- at most $r-1-j$ outgoing edges from vertex u_{i+j} for $1 \le j \le r-2$ which connect u_{i+j} to vertices $v_{i+j+1}, \dots, v_{\min\{i+r-1,p\}}$; see Fig. 11(c), and,
- at most $\ell - r - j - 1$ outgoing edges from vertex u_{i-j} for $1 \le j \le \ell - r - 2$ which connect u_{i-j} to vertices $v_{i+r+1}, \dots, v_{\min\{i+\ell-j,p\}}$; see Fig. 11(d).

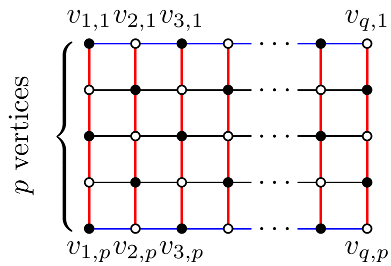

Figure 14. A graph ($p \times q$ grid) with pathwidth p.

In total, for the number of edges crossing (u_i, v_{i+r}) we have at most

$$r\ell + \sum_{i=1}^{\ell-1} i + \sum_{i=1}^{r-2} i + \sum_{i=1}^{\ell-r-2} i$$

$$= \ell r + \frac{\ell^2 - \ell}{2} + \frac{(r-2)(r-1)}{2} + \frac{(\ell - r - 2)(\ell - r - 1)}{2}$$

$$\le \ell r + \frac{\ell^2}{2} + \frac{r^2}{2} + \frac{(\ell - r)^2}{2} = \ell^2 + r^2.$$

The last term is maximal for $r = \ell$, yielding at most $2\ell^2$ crossings on $(u_i, v_{i+\ell})$. To ensure k-planarity we set $2\ell^2 \le k$ and obtain $\ell \le \sqrt{k/2}$, which implies that every vertex except for those with the $\ell = \mathcal{O}(\sqrt{k})$ largest indices has $\ell = \sqrt{k/2}$ outgoing edges. In total, the vertices with the largest ℓ indices are incident to $2 \cdot \sum_{i=0}^{\ell-1} i = \ell^2 - \ell = 2\ell^2 - \mathcal{O}(\ell^2)$ outgoing edges. The statement follows. ■

5. PROPERTIES OF 2-LAYER k-PLANAR GRAPHS

In this section, we present some properties of 2-layer k-planar graphs.

In Theorem 3.3, we have established that every optimal 2-layer 3-planar graph is (3-)quasiplanar, which is also the case in the general, non-layered, drawing model [21]. A more general relationship between the classes of k-planar and h-quasiplanar graphs was uncovered in [20], where it is proven that every k-planar graph is $(k+1)$-quasiplanar, for every $k \ge 2$. Next, we show that for 2-layer drawings an even stronger relationship holds.

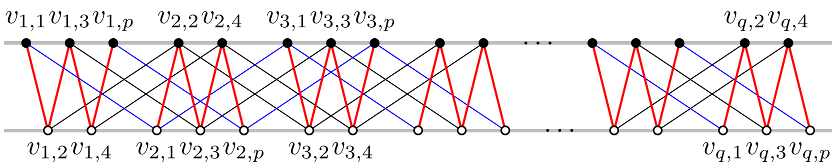

Figure 15. $(2p-3)$-planar drawing of the graph from Fig. 14. Note that $p = 5$.

Theorem 5.1. For $k \geq 3$, every 2-layer k-planar graph is 2-layer $\lceil \frac{2}{3}k + 2 \rceil$-quasiplanar. Further, every 2-layer 2-planar graph is 2-layer (3-)quasiplanar.

Proof. Let G be a topological 2-layer k-planar graph, with $k \geq 3$. In the following text, we assume G to be connected as otherwise we could apply the same argument independently on each connected component and place them one aside each other without crossing. Suppose for a contradiction that G contains $h := \lceil \frac{2}{3}k+2 \rceil$ mutually crossing edges (u'_i, v'_{h+1-i}) for $1 \leq i \leq h$ in G, such that $u'_1, \ldots, u'_h$ and $v'_1, \ldots, v'_h$ appear in this order in $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$, respectively. Observe that (u'_1, v'_h) and (u'_h, v'_1) have $h-1$ crossings from this h-tuple. Moreover, both endvertices of all the $h-2$ edges (u'_i, v'_{h+1-i}), for $i = 2, \ldots, h-1$, are located in regions bounded by $e^{(1)} := (u'_1, v'_h)$ and $e^{(2)} := (u'_h, v'_1)$; see Fig. 12. Since G is connected, for each $2 \leq i \leq h-1$, the edge (u'_i, v'_{h+1-i}) is adjacent to another edge e_i. Note that either $e_i = e_j$ for some $j \neq i$, and e_i crosses $e^{(1)}$ and $e^{(2)}$, or $e_i \neq e_j$ for all $j \neq i$, and e_i crosses one of $e^{(1)}$ and $e^{(2)}$. This implies $h-2$ additional crossings for $\{e^{(1)}, e^{(2)}\}$, and, consequently, $e^{(1)}$ or $e^{(2)}$ is crossed by at least $h - 1 + \lceil (h-2)/2 \rceil$ edges. We obtain $h - 1 + \lceil (h-2)/2 \rceil \geq \frac{3}{2}h - 2 \geq \frac{3}{2}\left(\frac{2}{3}k + 2\right) - 2 = k+1$ crossings for $e^{(1)}$ or $e^{(2)}$, a contradiction.

For the case $k = 2$, assume that G contains three mutually crossing edges $e_1 = (u'_1, v'_3)$, $e_2 = (u'_2, v'_2)$ and $e_3 = (u'_3, v'_1)$, such that u'_1, u'_2, u'_3 and v'_1, v'_2, v'_3. appear in this order in $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$, respectively. As e_1 and e_3 are already crossed twice, e_2 represents a connected component; contradiction. ■

Next, we show that the pathwidth of 2-layer k-planar graphs is bounded by $k + 1$. We point out that similar results are known for layered graphs with a bounded total number of crossings [50] and for layered fan-planar graphs [49], and that these bounds do not have any implication on 2-layer k-planar graphs.

Theorem 5.2. Every 2-layer k-planar graph has pathwidth at most $k + 1$.

Proof. Let G be a topological 2-layer k-planar graph with parts U and V. We first define a total ordering $\prec$ on the edges as follows: We say that edge $e_1 = (u_i, v_x)$ precedes edge $e_2 = (u_j, v_y)$, or $e_1 \prec e_2$, if $u_i, u_j \in U$ and either

i $i < j$, or
ii $i = j$ and $x < y$.

Let $E = (e_1, \ldots, e_m)$ be the set of edges ordered with respect to $\prec$. Let $e_i = (u_s, v_t)$ be an edge and let v_y be a vertex in V. Further let $f(y)$ and $\ell(y)$ denote the indices of the first and the last edge in $\prec$ incident to v_y, respectively, i.e. $e_{f(y)}$ and $e_{\ell(y)}$ are the first and the last edge incident to v_y in $\prec$, respectively. We call $v_y \in V$ *related* to e_i if v_y is incident to an edge crossing e_i and $f(y) < i < \ell(y)$. Refer to Fig. 13(a) for an illustration of an edge e_i with two related vertices v_y and $v_{y'}$ incident to edges crossing e_i. For every edge $e_i = (u_s, v_t) \in E$, we construct a bag B_i that contains u_s, v_t and all the (at most k) related vertices of e_i. Then, we connect B_i to bags B_{i-1} and B_{i+1} (if they exist), obtaining a path of bags P.

In the following text we show that P is a valid path decomposition of G. Since we assigned at most $k + 2$ vertices to each bag of P the width of P is at most $k + 1$. Properties P.1 and P.3 of a tree decomposition are fulfilled for P by construction. We may assume that G is connected, otherwise we compute a path decomposition for each connected component and link the obtained vertex disjoint paths. Hence also P.2 is fulfilled. Moreover, by the choice of $\prec$, all the edges incident to a vertex $u_i \in U$ occur in a consecutive sequence, i.e. u_i is incident to edges $e_j, \ldots, e_k$ for some $1 \leq j \leq k \leq m$ and then u_i appears in all of bags $B_j, \ldots, B_k$, which is a subpath of P. Therefore, Property P.4 also holds for all vertices in U.

It remains to show that Property P.4 holds for every vertex $v_y \in V$. Let $e_{f(y)} = (u_f, v_y)$ and $e_{\ell(y)} = (u_\ell, v_y)$. Note that each of the edges $e_{f(y)}, e_{f(y)+1}, \ldots, e_{\ell(y)}$ is either incident to v_y (see Fig. 13(b)), or it crosses one of $e_{f(y)}$ and $e_{\ell(y)}$, since its endvertex in U is some u_i with $f \leq i \leq \ell$; see Fig. 13(c) to (e). Note that for the endvertex v_z in V necessarily $z > y$ if $u_i = u_f$ or $z < y$ if $u_i = u_\ell$ by definition of $\prec$; see Fig. 13(d) or Fig. 13(e), respectively. Hence v_y belongs to all bags $B_{f(y)}, B_{f(y)+1}, \ldots, B_{\ell(y)}$ and P.4 holds. The statement follows. ■

Theorem 5.3. There exist topological 2-layer k-planar graphs with pathwidth $p = \frac{k+3}{2}$.

Proof. Consider the $p \times q$ grid graph G_p with $p < q$; see Fig. 14. There is a natural labeling of the vertices of G_p according to the grid structure as follows. Namely, we can label the vertices with two indices $x \in \{1, \ldots, q\}, y \in \{1, \ldots, p\}$ so that $v_{x,y}$ is incident exactly to $v_{x,y+1}$ (unless $y = p$), $v_{x,y-1}$ (unless $y = 1$), $v_{x+1,y}$ (unless $x = q$) and $v_{x-1,y}$ (unless $x = 1$). Since G_p contains the $p \times p$ grid, its treewidth is at least p. Since it is also possible to find a path decomposition of G_p with width p, we conclude that $pw(G_p) = p$.

Now consider the 2-layer graph Γ_p of Fig. 15 whose underlying graph is G_p. Namely, for $x \in \{1, \ldots, q\}$, the subgraph induced by vertices $v_{x,1}, v_{x,2}, \ldots, v_{x,p}$ is realized as a planar path of length $p-1$ (see red edges in Fig. 15), where $v_{x,i}$ precedes $v_{x,j}$ along L_u or L_v if and only if $j = i + 2k$ for $k \geq 0$. Further, the subdrawings Γ_x and $\Gamma_{x'}$ realizing subgraphs induced by vertices $v_{x,1}, v_{x,2}, \ldots, v_{x,p}$ and vertices $v_{x',1}, v_{x',2}, \ldots, v_{x',p}$, respectively, are disjoint, with all vertices of Γ_x preceding all vertices of $\Gamma_{x'}$ if and only if $x < x'$.

Now we compute the maximum number of crossings per edge. Each edge $(v_{x,y}, v_{x,y+1})$ (see red edges in Fig. 15) is crossed by at most one edge incident to each of $v_{x,y-1}, v_{x,y-2}, \ldots, v_{x,1}$ and by at most one edge incident to each of $v_{x-1,y+2}, v_{x-1,y+3}, \ldots, v_{x-1,p}$ (i.e. at most $p-2$ times). On the other hand, each edge $(v_{x,y}, v_{x+1,y})$ (see blue and black edges in Fig. 15) is crossed by at most one of $(v_{x-1,y'}, v_{x,y'})$, $(v_{x,y'}, v_{x+1,y'})$ and $(v_{x+1,y'}, v_{x+2,y'})$ for each $y' \neq y$ (the exact edge depends on the parities of y and y').

This amounts to at most $p-1$ crossings. Moreover, such edges are crossed by edges $(v_{x,y+1}, v_{x,y+2}), (v_{x,y+2}, v_{x,y+3}), \ldots, (v_{x,p-1}, v_{x,p})$

and $(v_{x+1,y-1}, v_{x+1,y-2}), \ldots, (v_{x+1,2}, v_{x+1,1})$, i.e. at most $p-2$ additional times. We conclude that edges $(v_{x,1}, v_{x+1,1})$ and $(v_{x,p}, v_{x+1,p})$ (see blue edges in Fig. 15) are crossed most often, namely $2p-3$ times. Thus, Γ_p is $2p-3$-planar which concludes the proof. ■

6. CONCLUSIONS

We gave results for 2-layer k-planar graphs regarding their density, relationship to 2-layer h-quasiplanar graphs and pathwidth. Tight density bounds for 2-layer k-planar graphs with $k = 6$ may be achievable following similar arguments to the proof of Theorem 3.5, which would also improve upon our results for the Crossing Lemma, and in turn on the density for general k. Moreover, a better lower bound for general k may exist. The relationship to other beyond-planar graph classes is also of interest. With respect to the pathwidth, we conjecture that our upper bound is tight. Finally, while we managed to characterize optimal 2-layer 2- and 4-planar graphs, the general recognition and characterization of 2-layer k-planar graphs remain important open problems.

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